

THE MINIMAL DEGREE OF A FINITE INVERSE SEMIGROUP

BORIS M. SCHEIN

ABSTRACT. The minimal degree of an inverse semigroup S is the minimal cardinality of a set A such that S is isomorphic to an inverse semigroup of one-to-one partial transformations of A . The main result is a formula that expresses the minimal degree of a finite inverse semigroup S in terms of certain subgroups and the ordered structure of S . In fact, a representation of S by one-to-one partial transformations of the smallest possible set A is explicitly constructed in the proof of the formula. All known and some new results on the minimal degree follow as easy corollaries.

If ρ is an isomorphic or a homomorphic representation of an inverse semigroup S by one-to-one partial transformations of a set A , then the cardinality of A is denoted by $\delta(\rho)$ and called the *degree* of ρ . Every inverse semigroup has a faithful (that is, isomorphic) representation by one-to-one partial transformations of a set. The minimal degree of a faithful representation of S is called the *minimal degree of S* and denoted by $\delta(S)$. If the generalized continuum hypothesis is assumed and S is infinite, then $\delta(S)$ is either $|S|$ or the predecessor of $|S|$. This follows from the fact that \mathcal{S}_A , the symmetric inverse semigroup of all one-to-one partial transformations of an infinite set A , has cardinality $2^{|A|}$. In particular, if $|S|$ is a limit cardinal, the $\delta(S) = |S|$. While finding $\delta(S)$ for infinite S is not devoid of interest, we consider only finite inverse semigroups in this paper. Our main result is an exact formula for $\delta(S)$ “modulo groups.” Solving semigroup problems “modulo groups” (a semigroup problem reduced to a group problem is considered solved) may raise objections, but in our case a “modulo groups” solution may be the best one can expect. Indeed, if G is a group, then $\delta(G)$, as is easily seen (Lemma 1), is the minimal degree of faithful representations of G by permutations. The minimal degree of groups has been considered for more than a century, no definitive formula for $\delta(G)$ has been found yet, and one can hardly expect that such a formula exists.

If H is a subgroup of G , we define $\delta_G(H)$, the *minimal degree of H in G* , as the minimal cardinality of a *nonempty* set A such that there exists a (homomorphic) representation P of G by permutations of A which induces a faithful representation of H . Such a representation ρ is called a *minimal representation of H in G* . It seems that this concept has not been considered in the theory of groups, although it belongs to this theory. Observe that A above

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is not empty for the following reason. A trivial group is isomorphic to the group of all permutations of an empty set, and hence, from our point of view, its degree is 0 (and not 1!). Yet we want $\mu(D)$, as defined below, to be positive.

One may consider representations of an inverse semigroup S by partial transformations that are not necessarily one-to-one. As proved in [9], the minimal degree of S with respect to such representations coincides with $\delta(S)$.

To state our main result we introduce a few known and less known concepts.

Every inverse semigroup S is *naturally* (or *canonically*) ordered by a partial order relation ω , where $s\omega t$ means $s = ss^{-1}t$ for any $s, t \in S$. Here, as usual, s^{-1} is the inverse of s , that is, the only element of S such that $ss^{-1}s = s$ and $s^{-1}s = s^{-1}ss^{-1}$. Since an inverse semigroup is an ordered set, all usual concepts of the theory of ordered sets can be applied. In particular, an element of S may be the least upper bound (the l.u.b.) of a subset of S . An element $s \in S$ is *join-irreducible* if it is not the l.u.b. of any subset of S that does not contain s . For example, the zero element of S , if it exists, is the l.u.b. of the empty subset of elements of S , and hence the zero element is not join-irreducible. If $s < t$ (that is, $s\omega t$ and $s \neq t$), then s is called a *strict minorant* of t .

Green's relations \mathcal{D} and \mathcal{J} coincide on every finite semigroup. Thus elements s and t of an inverse semigroup S are \mathcal{D} -equivalent if and only if they generate the same principal ideal of S or, equivalently, if $ss^{-1} = uu^{-1}$ and $u^{-1}u = t^{-1}t$ for some $u \in S$. Each \mathcal{D} -class of a finite inverse semigroup S is a Brandt groupoid with respect to the partial multiplication induced on it. Each Brandt groupoid is isomorphic to a groupoid of the form $I \times G \times I$, where I is a nonempty set and G is a group. The product of its elements (i, g, j) and (k, h, l) is defined precisely when $j = k$, in which case it is equal to (i, gh, l) . Thus every \mathcal{D} -class D_s of S is isomorphic to a Brandt groupoid of the form $I \times G \times I$. Here G is isomorphic to any of the maximal subgroups of D_s , for example, $G \simeq H_{ss^{-1}}$, where $H_{ss^{-1}}$ is the \mathcal{H} -class of S that contains ss^{-1} (that is, $H_{ss^{-1}}$ is the maximal subgroup of S whose identity element is ss^{-1}). The cardinality of the set I equals the cardinality of the set of all idempotents of D_s because idempotents of $I \times G \times I$ are precisely the triples of the form (i, e, i) , where e is the identity element of G . Let $\iota(D)$ denote the cardinality of the set of all idempotents in a \mathcal{D} -class D .

A \mathcal{D} -class D is called *join-irreducible* if it contains a join-irreducible element. It is proved in Lemma 2 that all elements of a join-irreducible \mathcal{D} -class are join-irreducible. In the sequel \mathbf{D} denotes the set of all join-irreducible \mathcal{D} -classes of S .

Let H_e be the maximal subgroup of S with e as its identity element. Define the following subset S_e of H_e : $h \in S_e$ if and only if $f \leq h$ for all f in S such that $f < e$. Thus $h \in S_e$ if $hh^{-1} = h^{-1}h = e$ and e and h have precisely the same strict minorants in S . For example, if $h \in S_e$ and $h \neq e$, then e and h have the same strict minorants, and hence e is not the l.u.b. of the set of its strict minorants. Thus, if $S_e \neq \{e\}$, then e is join-irreducible. It is proved in Lemma 3 that S_e is a normal subgroup of H_e .

Let $e \in D$, where D is a join-irreducible \mathcal{D} -class of S . Define $\mu(D) = \delta_{H_e}(S_e)$. We prove in Lemma 4 that $\mu(D)$ is well defined, that is, it does not depend on the choice of e in D . It follows from our definition of $\delta_G(H)$ that $\mu(D) \geq 1$. Obviously, $\mu(D) = 1$ if and only if S_e is a trivial group.

Now we are ready to state our main result.

Theorem. *If S is a finite inverse semigroup, then*

$$\delta(S) = \sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\}.$$

Proof. Our equality follows from Lemmas 6 and 7.

Lemma 1. *If G is a group, then $\delta(G)$ is the minimal degree of faithful representations of G by permutations.*

Proof. Let R be a representation of G by one-to-one partial transformations of a set A . Then $R(e)$ is the identity mapping of a subset B of A . For every $g \in G$, $gg^{-1} = e$ and $eg = g$. It follows that $R(g)R(g^{-1}) = R(e)$ and $R(e)R(g) = R(g)$. Therefore, the domains of $R(g)$ and $R(e)$ coincide, that is, B is the domain of $R(g)$. Analogously, B is the range of $R(g)$. Thus $R(g)$ is a permutation of B . Restricting $R(g)$ to B for all $g \in G$ we obtain a representation R_B by permutations of B . If R is faithful, so is R_B , and hence if R is minimal (that is, it is faithful and $\delta(G) = |A|$), then $B = A$ and R is a representation of G by permutations of a set. \square

Lemma 2. *All elements of a join-irreducible \mathcal{D} -class of an inverse semigroup are join-irreducible.*

Proof. Let $s, t \in D$. There are $u, v \in D$ such that $s = utv$ and $t = u^{-1}sv^{-1}$. If t is the l.u.b. of elements t_i , $1 \leq i \leq n$, then $t_i \leq t$, and so $s_i = ut_i v \leq utv = s$ for all i . Let $s_i \leq w$ for all i . Then $t_i = u^{-1}ut_i vv^{-1} \leq u^{-1}s_i v^{-1} \leq u^{-1}wv^{-1}$ for all i , so that $t \leq u^{-1}wv^{-1}$. Now $s = utv \leq uu^{-1}wv^{-1}v \leq w$. Thus s is the l.u.b. of elements s_i , $1 \leq i \leq n$. If s is join-irreducible, $s = s_i$ for some i . It follows that $t = u^{-1}sv^{-1} = u^{-1}s_i v^{-1} = t_i$, and hence t is join-irreducible. \square

Lemma 3. *S_e is a normal subgroup of H_e .*

Proof. Let $u, v \in S_e$. If $f < e$, then $f < u$ and $f < v$. Since f is idempotent, $f < v^{-1}$, and hence $f = f^2 \leq uv^{-1}$, where $uv^{-1} \in H_e$. Thus $f \neq uv^{-1}$, so that $f < uv^{-1}$. It follows that $uv^{-1} \in S_e$. Also, $e \in S_e$, so that $S_e \neq \emptyset$. Therefore, S_e is a subgroup of H_e .

If $u \in S_e$ and $v \in H_e$, then $f < u$ for every f such that $f < e$. It follows that $vf v^{-1} \leq vev^{-1} = e$. If $vf v^{-1} = e$, then $f = efe = v^{-1}vf v^{-1}v = v^{-1}ev = e$, and hence $vf v^{-1} < e$. Therefore, $vf v^{-1} < u$, so that $f = v^{-1}vf v^{-1}v < v^{-1}uv$. Thus $v^{-1}uv \in S_e$ and S_e is a normal subgroup of H_e . \square

Lemma 4. *$\mu(D)$ does not depend on the choice of e in D .*

Proof. Let e and f be idempotents in a join-irreducible \mathcal{D} -class D . Then the groups H_e and H_f are isomorphic, for there exists $v \in D$ such that the mapping $i^v(x) = v^{-1}xv$ for all $x \in H_e$ is an isomorphism of H_e onto H_f . To see that $i^v(S_e) = S_f$, assume that $u \in S_e$. If $k < f$, then $vk v^{-1} < vfv^{-1} = e$, so that $vk v^{-1} < u$. Thus $k = f k f = v^{-1}vk v^{-1}v < v^{-1}uv = i^v(u)$ and $i^v(u) \in S_f$, i.e., i^v maps S_e into S_f . The inverse isomorphism of S_f onto S_e maps S_f into S_e , so that i^v maps S_e onto S_f , and hence groups S_e and S_f are isomorphic. \square

The reader is supposed to be familiar with the theory of representations of inverse semigroups by one-to-one partial transformations of sets as developed in [6]. Those readers who are not as fluent in Russian as they undoubtedly wish they were, can find an exposition of this theory in English in Chapter 7 of [1], Chapter V of [4], or Chapter IV of [5]. To make this paper reasonably self-contained, we give a very brief description of some concepts of this theory here.

A subset T of an inverse semigroup S is called *majorantly closed* if $s \in T$ and $s \leq t$ imply $t \in T$ for all $s, t \in S$. If U is a subset of S , then \overline{U} denotes its majorant closure, that is, $t \in \overline{U}$ if and only if $s \leq t$ for some $s \in U$. If F is an inverse subsemigroup of S , its majorant closure \overline{F} is a majorantly closed inverse subsemigroup of S .

Let T be a majorantly closed inverse subsemigroup of S , $s \in S$ and $ss^{-1} \in T$. The subset \overline{Ts} of S is called a *right coset* of T . If $ss^{-1} \notin T$, the right coset of T is not defined. The set of all right cosets of T is denoted by S/T . If $s^{-1}s \in T$, then \overline{sT} is called a *left coset* of T . Right (left) cosets of T either are disjoint or coincide, and they need not cover S . The number of right cosets of T is called the *index* of T in S and denoted by $[S : T]$. Thus $[S : T] = |S/T|$. If \overline{Ts} is a right coset of T , then $\overline{s^{-1}T}$ is a left coset. This gives a one-to-one correspondence between right and left cosets of T , so that $[S : T]$ is the number of left cosets of T .

Let T be a majorantly closed inverse subsemigroup of an inverse semigroup S . The transitive representation of S by one-to-one partial transformations of the set S/T is denoted by P_T . For every $s \in S$ and every right coset \overline{Tt} of T , \overline{Tt} belongs to the domain of $P_T(s)$ precisely if the right coset \overline{Tts} exists (that is, if $(ts)(ts)^{-1} \in T$), in which case the image of \overline{Tt} under $P_T(s)$ is \overline{Tts} . Thus $\delta(P_T) = [S : T]$. Also $P_T(s) = P_T(t)$ for $s, t \in S$ if and only if $(\forall x, y \in S)[xsy \in T_i \Leftrightarrow xty \in T_i]$.

If S is finite, every majorantly closed inverse subsemigroup T of S is a majorant closure of a subgroup of S . Indeed, if e is the least idempotent of T and F the maximal subgroup of T that contains e , then $\overline{F} \subset T$. If $h \in T$, then $ehe \in T$. It follows from $e(ehe) = (ehe)e = ehe$ that $ehe \in F$. The inequality $ehe \leq h$ shows that $h \in \overline{F}$. Thus $T = \overline{F}$.

Lemma 5. Let $T = \overline{F}$ be a majorantly closed inverse subsemigroup of an inverse semigroup S , where F is a maximal subgroup of T . Let G be the maximal subgroup of S that contains F , and D the \mathcal{D} -class of S that contains G . Then $\delta(P_T) = [G : F] \cdot \iota(D)$.

Proof. Clearly, $\delta(P_T) = [S : T]$. Let $\overline{Ts} \in S/T$. Then $Fs \subset Ts$, and if $ts \in Ts$ for $t \in T$, then $g \leq t$ for some $g \in F$ and $gs \leq ts$. Thus $\overline{Ts} = \overline{Fs}$. Since $ss^{-1} \in T$, we see that $(es)(es)^{-1} = ess^{-1}e = e$, and hence $es \in D$ and $Fes = (Fe)s = Fs$. Therefore, all right cosets of T have the form $\overline{Tu} = \overline{Fu}$ with $u \in D$. If $\overline{Tu} = \overline{Tv}$ for $u, v \in D$, then $uv^{-1} \in T$, so that $uu^{-1}, vv^{-1} \in T$ and $uu^{-1} = vv^{-1} = e$. If $g \leq uv^{-1}$ for $g \in F$, then $gv \leq uvv^{-1} \leq u$. Now,

$$(gv)(gv)^{-1} = gvv^{-1}g^{-1} = geg^{-1} = e = uu^{-1},$$

so that $gv = u$ or, equivalently, $\overline{Fu} = \overline{Fv}$. Conversely,

$$\overline{Fu} = \overline{Fv} \Rightarrow uv^{-1} \in F \Rightarrow uv^{-1} \in T \Rightarrow \overline{Tu} = \overline{Tv}.$$

Thus $\overline{Tu} = \overline{Tv}$ if and only if $\overline{Fu} = \overline{Fv}$. It follows that the number of right cosets of T in S equals the number of right cosets of F in D .

Since D is finite, it is a Brandt groupoid under the partial multiplication induced by that of S . Therefore, D is isomorphic to a Brandt groupoid of the form $I \times K \times I$, where I is a nonempty set, K is a group, the product of triples (i, g, j) and (k, h, l) is defined precisely when $j = k$, and $(i, g, j)(j, h, l) = (i, gh, l)$. The group K is isomorphic to any maximal subgroup of the groupoid, and so we assume that $K = G$. Idempotents of $I \times K \times I$ have the form (i, e, i) , where e is the identity element of K , and hence there are precisely as many idempotents in D as there are elements in I , so that $\iota(D) = |I|$. Every subgroup of $I \times G \times I$ is of the form $M = \{i\} \times F \times \{i\}$, where F is a subgroup of G and $i \in I$. If $u \in I \times G \times I$, then $uu^{-1} \in M$ means that $u = (i, g, j)$ for some $g \in G$ and $j \in I$. Also, $Mu \neq \emptyset$ if and only if $u = (i, g, j)$ for some $g \in G$ and $j \in I$, so that $Mu = \{i\} \times Fg \times \{j\}$. Thus, if $u, v \in I \times G \times I$ and both Mu and Mv are not empty, then $Mu = Mv$ exactly if $u = (i, g, j)$ and $v = (i, h, j)$ for some $g, h \in G$ and $j \in I$, in which case $Fg = Fh$. Therefore, the number of different nonempty sets Mu is the number of different right cosets of F in G multiplied by the number of all elements j in I . In other words, this number is the index of F in G multiplied by the cardinality of I , that is, $[G : F] \cdot \iota(D)$. The number of Mu in $I \times G \times I$ equals the number of Fu in D , which, in its own turn, equals the number of right cosets of T in S . \square

Lemma 6. $\delta(S) \leq \sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\}$.

Proof. We will construct a representation P of S . Choose a collection of subgroups of S in the following way. From each $D \in \mathbf{D}$ choose an idempotent $e \in D$ and consider the group S_e . Let R_e be a minimal representation of S_e in H_e by permutations. Then R_e is a sum of n_e transitive representations of H_e , and each of these representations is similar to a representation R_i by permutations of right cosets of a subgroup F_i of H_e , $i = 1, 2, \dots, n_e$. The degree $\delta(R_e)$ of R_e is the sum of degrees of these representations R_i , and thus it is the sum of indexes of F_i in H_e . Let $T_i = \overline{F}_i$. The transitive representation P_{T_i} of S by one-to-one partial transformations of the set S/T_i of right cosets of T_i will be denoted by P_i . By Lemma 5, $\delta(P_i) = \iota(D) \cdot [S : T_i]$. Let P_e be the sum of representations P_i , $1 \leq i \leq n_e$. Then

$$\delta(P_e) = \sum \{\delta(P_i) : 1 \leq i \leq n_e\} = \sum \{\iota(D) \cdot [S : T_i] : 1 \leq i \leq n_e\} = \iota(D) \cdot \mu(D).$$

Let P be the sum of P_e for all $D \in \mathbf{D}$. Then $\delta(P) = \sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\}$. If P is faithful, then $\delta(S) \leq \delta(P)$. Thus it suffices to prove that P is faithful. We do that in a few steps. It suffices to prove the implication $P(s) = P(t) \Rightarrow s \leq t$ for $s, t \in S$. Indeed, if it holds and $P(s) = P(t)$, then $s \leq t$ and, interchanging s and t , we obtain $t \leq s$, so that $s = t$ and P is faithful.

Therefore, assume that $P(s) = P(t)$. If $z \leq s$ for no join-irreducible elements $z \in S$, then s is the join of an empty set of join-irreducible elements, i.e. $s = 0$, so that $s \leq t$. Let $z \leq s$ for some join-irreducible element $z \in S$. Our goal is to prove that $z \leq t$. Since $z \leq s$, we obtain $z = zz^{-1}s$, and hence

$$P(z) = P(zz^{-1}s) = P(zz^{-1})P(s) = P(zz^{-1})P(t) = P(zz^{-1}t).$$

Let D be the \mathcal{D} -class of S that contains z . Obviously, $D \in \mathbf{D}$. Let e be the idempotent of D chosen in our construction of P . Clearly, $P_e(z) = P_e(zz^{-1}t)$. Since P_e is the sum of representations P_i , $i = 1, \dots, n_e$, we obtain $P_i(z) = P_i(zz^{-1}t)$ for all i . This means that $(\forall x, y \in S)[xzy \in T_i \Leftrightarrow xzz^{-1}ty \in T_i]$.

Let $v^{-1}v = e$ and $vv^{-1} = z^{-1}z$, so that $v \in D$. If $x = (zv)^{-1}$ and $y = v$, then

$$(zv)^{-1}zv \in T_i \Leftrightarrow (zv)^{-1}zz^{-1}tv \in T_i.$$

But $(zv)^{-1}zv = e \in T_i$, and hence $v^{-1}z^{-1}tv = (zv)^{-1}zz^{-1}tv \in T_i$. Clearly,

$$v^{-1}z^{-1}tv = e(v^{-1}z^{-1}tv)e \in eT_ie = F_i,$$

where F_i is a maximal subgroup of T_i such that $e \in F_i$. A straightforward argument shows that $R_i(v^{-1}z^{-1}tv) = R_i(e)$ for all v such that $v^{-1}v = e$ and $vv^{-1} = z^{-1}z$, where R_i is a representation of the group H_e by permutations of the right cosets of F_i in H_e . This holds for all $i = 1, \dots, n_e$, and hence $R_e(v^{-1}z^{-1}tv) = R_e(e)$ for all v such that $v^{-1}v = e$ and $vv^{-1} = z^{-1}z$.

To prove that $z \leq t$ we use an inductive procedure over join-irreducible idempotents e .

(1) If e is a minimal element in the set of all join-irreducible idempotents, then $S_e = H_e$. Indeed, the implication $f < e \Rightarrow f < g$ holds for all $g \in H_e$ and all join-irreducible idempotents f , because $f < e$ is always false. Since R_e is a minimal representation of the group S_e in H_e and $S_e = H_e$, we see that R_e is a minimal representation of H_e by permutations, and hence R_e is faithful. It follows that $uzv = uz z^{-1}tv$. Therefore, $z = u^{-1}uzvv^{-1} = u^{-1}uz z^{-1}tvv^{-1} \leq t$.

(2) Let $z \leq s \Rightarrow z \leq t$ for all $z \in D_f$ and all idempotents f such that $D_f < D_e$. Here $D_f < D_e$ means that $D_f \neq D_e$ and $D_f \leq D_e$ in the sense of the usual ordering on \mathcal{D} -classes of S . Let g be a join-irreducible idempotent such that $g < e$. Then $D_g < D_e$ and, since vgv^{-1} is idempotent, $z(vgv^{-1}) \leq z \leq s$, so that

$$\begin{aligned} [z(vgv^{-1})]^{-1}[z(vgv^{-1})] &= (vgv^{-1})z^{-1}z(vgv^{-1}) \\ &= (vgv^{-1})vv^{-1}(vgv^{-1}) = (vg)(vg)^{-1} \end{aligned}$$

and $(vg)^{-1}(vg) = gv^{-1}vg = geg = g$. Therefore, $z(vgv^{-1})^{-1} \in D_g$, and hence

$$z(vgv^{-1}) \leq t \quad \text{and} \quad [z(vgv^{-1})]^{-1}t = [z(vgv^{-1})]^{-1}[z(vgv^{-1})] = vgv^{-1}.$$

Thus

$$\begin{aligned} gv^{-1}z^{-1}tv &= g(v^{-1}v)v^{-1}z^{-1}tv = (v^{-1}v)gv^{-1}z^{-1}tv \\ &= v^{-1}[z(vgv^{-1})]^{-1}tv = v^{-1}vgv^{-1}v = geg = g. \end{aligned}$$

It follows that $g \leq v^{-1}z^{-1}tv$. However, $v^{-1}z^{-1}tv \in H_e$ and $g \notin H_e$, so that $g < v^{-1}z^{-1}tv$.

Thus $g < v^{-1}z^{-1}tv$ for all join-irreducible g such that $g < e$. Let f be an idempotent such that $f < e$. Then $D_f < D_e$. Obviously, each element of S is a join of join-irreducible elements. Let f be a join of join-irreducible elements g_1, \dots, g_k . Clearly, $g_i \leq f < e$, and hence $g_i < e$ and g_i is a join-irreducible idempotent for every $i = 1, \dots, k$. It follows that $g_i < v^{-1}z^{-1}tv$

for all i , and hence $f < v^{-1}z^{-1}tv$. Thus $v^{-1}z^{-1}tv \in H_e$, and if $f < e$ then $f < v^{-1}z^{-1}tv$. Therefore, $v^{-1}z^{-1}tv \in S_e$.

We proved earlier that $R_e(v^{-1}z^{-1}tv) = R_e(e)$. Now we see that both $v^{-1}z^{-1}tv$ and e belong to S_e . Since R_e is a minimal representation of S_e in H_e , it is faithful on S_e . Thus $v^{-1}z^{-1}tv = e$, and hence

$$z = zz^{-1}zz^{-1}z = zvv^{-1}vv^{-1} = zvev^{-1} = zvv^{-1}z^{-1}tvv^{-1} \leq t.$$

It follows that $z \leq t$.

We proved that $z \leq s \Rightarrow z \leq t$ for all join-irreducible z in S . If s is the l.u.b. of join-irreducible elements z_1, z_2, \dots, z_k , then $z_i \leq s$ for all $i = 1, 2, \dots, k$, and hence $z_i \leq t$ for all i . Therefore, $s \leq t$. \square

Lemma 7. $\sum\{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\} \leq \delta(S)$.

Proof. Let R be a faithful representation of S by one-to-one partial transformations of a finite set. Then R is a sum of transitive representations and every transitive representation is similar to P_T , where T is a majorantly closed inverse subsemigroup of S (see [6]). We can assume that R is a sum of a family $\{P_{T_i}\}_{i \in I}$ of transitive representations. Also, assume that $T_i = \overline{F_i}$, where F_i is a subgroup of T_i . Let e_i denote the identity element of F_i , and let e_i belong to a \mathcal{D} -class D_i of S .

Suppose that $D \in \mathbf{D}$ and $e^2 = e \in D$. Let $s \in S$ be such that $x^{-1}e_ix \leq e \Rightarrow x^{-1}e_ix \leq s$ for all $i \in I$ and $x \in S$ such that $x^{-1}e_ix \in D_i$. If $xey \in T_i$ for some $i \in I$, then $xey \leq xy$ and $(xey)^{-1}(xey) \leq y^{-1}ey$, so that $xy, y^{-1}ey \in T_i$. Also, $y^{-1}ey \in T_i$ implies $e_i \leq y^{-1}ey \leq y^{-1}y$. It follows that $ye_iy^{-1} \in D_i$ and $ye_iy^{-1} \leq yy^{-1}e_yy^{-1} \leq e$. By our assumption, $ye_iy^{-1} \leq s$, so that $e_i = y^{-1}ye_iy^{-1}y \leq y^{-1}sy$ and $y^{-1}sy \in T_i$. Clearly, $(xy)(y^{-1}sy) \leq xsy$, and so $xsy \in T_i$. Thus $(\forall i \in I)(\forall x, y \in S)[xey \in T_i \Rightarrow xsy \in T_i]$, which implies $R(e) \subset R(s)$ (see [6]). Since R is faithful, we obtain $e \leq s$. It follows that e is the l.u.b. of elements of the form $x^{-1}e_ix$ for all $x \in S$ such that $x^{-1}e_ix \in D_i$. Since e is join-irreducible, we have $e = x^{-1}e_ix \in D_i$ for some $i \in I$, and so $D = D_i$. Let $I_D = \{i \in I : D = D_i\}$.

Suppose that $D_i = D_j$. Then $e_i = uu^{-1}$ and $e_j = u^{-1}u$ for some $u \in S$. The representation P_{T_i} is similar to the representation corresponding to the majorantly saturated inverse subsemigroup $\overline{u^{-1}T_iu}$ (see [6]), and so we can replace P_{T_i} by this similar representation. It follows from $T_i = \overline{F_i}$ that $\overline{u^{-1}T_iu} = \overline{u^{-1}F_iu}$. But $e_j = u^{-1}e_iu \in \overline{u^{-1}F_iu}$. Thus, without loss of generality, we can assume that $e_i = e_j$ for all $i, j \in I$ such that $D_i = D_j$.

Choose $D \in \mathbf{D}$ and consider a sum R_D of all representations P_{T_i} , $i \in I_D$. All idempotents $e_j \in D$ considered above are equal, and so we skip the index j and write just $e_j = e$. Restricting R_D to the subgroup H_e of S , we obtain a representation P_D of H_e that is the sum of transitive representations of H_e corresponding to all subgroups $F_i \subset D$. Let $h \in S_e$ and $P_D(h) = P_D(e)$. Then $R_D(h) = R_D(e)$ and $P_{F_i}(h) = P_{F_i}(e)$ for all $i \in I_D$, where P_{F_i} denotes the representation of H_e over the right cosets of its subgroup F_i . Let $xey \in T_i$ for some $i \in I$. Since $xey \leq xy$ and $(xey)^{-1}(xey) \leq y^{-1}ey$, we obtain $xy, y^{-1}ey \in T_i$. Thus $e_i \leq y^{-1}ey$. As we have seen above, $ye_iy^{-1} \leq e$, where $ye_iy^{-1} \in D_i$. If $ye_iy^{-1} < e$, then $ye_iy^{-1} < h$, because $h \in S_e$. Then $e_i = y^{-1}ye_iy^{-1}y \leq y^{-1}hy$, and so $y^{-1}hy \in T_i$. Since $(xy)(y^{-1}hy) \leq xhy$, we obtain $xhy \in T_i$. Now let $ye_iy^{-1} = e$. Then $e \in D_i$, $D_i = D$, and $e_i = e$. It

follows from $yey^{-1} = e$ that

$$y^{-1}e = y^{-1}yey^{-1} = ey^{-1}yy^{-1} = ey^{-1} \in H_e, \quad \text{and} \\ ey^{-1}y = (ye)^{-1}y = (ey)^{-1}(ey) = e,$$

so that $(F_i ey^{-1})e = (F_i y^{-1})e = (F_i ey^{-1})$ for every $i \in I_D$. Since $P_D(h) = P_D(e)$, we obtain $(F_i ey^{-1})h = (F_i ey^{-1})$. Thus

$$F_i y^{-1}hy = (F_i e)y^{-1}hy = (F_i ey^{-1})hy = (F_i ey^{-1})y = F_i(ey^{-1}y) = F_i e = F_i$$

and $y^{-1}hy \in F_i \subset T_i$. Therefore, $xhy \in T_i$.

We proved that $(\forall i \in I)(\forall x, y \in S)[(xey \in T_i \Rightarrow xhy \in T_i)]$. Therefore (see [6]), $R(e) \subset R(h)$. Since R is faithful, we obtain $e \leq h$, so that $e = eh = h$. Thus P_D is faithful on S_e . Then

$$\mu(D) = \delta_{H_e}(S_e) \leq \delta(P_D) = \sum \{[H_e : F_i] : i \in I_D\}.$$

By Lemma 5,

$$\iota(D) \cdot \mu(D) \leq \sum \{\iota(D) \cdot [H_e : F_i] : i \in I_D\} = \delta(R_D),$$

and hence

$$\sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\} \leq \sum \{\delta(R_D) : D \in \mathbf{D}\} \leq \delta(R)$$

for every faithful representation R of S . Thus $\sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\} \leq \delta(S)$. \square

Using the Theorem, we may easily obtain all known results on $\delta(S)$. Consider the following example. A (finite) inverse semigroup is called *unruly* if $\delta(S) = |S|$. These inverse semigroups were described in [7]. We obtain this result as Corollary 1. First we need a few known definitions. A *semilattice of groups* is an inverse semigroup S that is a union of its maximal subgroups H_e , $e \in E$, where E is the semilattice of idempotents of S . If $e, f \in E$ and $f < e$, then the mapping $\varphi_{e,f} : H_e \rightarrow H_f$ defined by $h\varphi_{e,f} = hf$ for every $h \in H_e$ is called a *structural homomorphism* of S . This structural homomorphism is called *trivial* if its range coincides with $\{f\}$, it is called *injective* if it is one-to-one. An idempotent e is called *isolated* if $H_e = \{e\}$.

Corollary 1 [7]. *A finite inverse semigroup S is unruly (that is, $\delta(S) = |S|$), if and only if S satisfies the following conditions:*

- (1) S is a semilattice of groups of the following types: (i) cyclic groups of prime power order; (ii) generalized quaternion groups; (iii) Klein's four-group; (iv) trivial group;
- (2) no structural homomorphisms are injective except those defined on trivial subgroups; all structural homomorphisms defined on Klein's four-groups are trivial;
- (3) S has no zero, and no isolated idempotent is a join of two of its strict minorants.¹

Proof. By our Theorem, S is unruly if and only if $\sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}\} = |S|$. Clearly, $\mu(D) \leq |H_e|$, where $e \in D$. Thus $\iota(D) \cdot \mu(D) \leq \iota(D)^2 \cdot |H_e| = |D|$, so that S is unruly if and only if $|D| = \iota(D) \cdot \mu(D)$ and every \mathcal{D} -class D of

¹This property corrects an inaccuracy in the statement of this result in [7].

S is join-irreducible, which, by Lemma 2, means that all idempotents of S are join-irreducible. The first condition means that $\iota(D)^2 \cdot |H_e| = \iota(D) \cdot \mu(D)$, that is, $\iota(D) \cdot |H_e| = \mu(D) \leq |H_e|$. Equivalently, $\mu(D) = |H_e|$, and $\iota(D) = 1$, so that D contains a single idempotent. Therefore, $D = H_e$ and S is a semilattice of groups H_e . Obviously, $\mu(D_e) \leq \delta(H_e)$ for $H_e \neq \{e\}$, and thus all nontrivial groups H_e are unruly. Amazingly, unruly groups do not seem to be described in the group literature. Their description was given in [7], and they are precisely the groups of types (i), (ii), and (iii). If $e, f \in E$, $f < e$, and $h \in S_e$, then $f < h$, and hence $e\varphi_{e,f} = ef = f = hf = h\varphi_{e,f}$. If $\varphi_{e,f}$ is injective, then $e = h$, and hence $S_e = \{e\}$. Then $|H_e| = \mu(H_e) = 1$, and hence H_e is trivial. Now suppose that H_e is a Klein four-group. As we have just seen, $S_e \neq \{e\}$. For $|S_e| = 2$ an easy computation shows that $\mu(H_e) = 2 \neq \delta(H_e)$. Therefore, $|S_e| = 4$, and hence $S_e = H_e$. This means that $\mu(H_e) = |H_e|$ and all structural homomorphisms $\varphi_{e,f}$ for $f < e$ are trivial. As the zero element is not join-irreducible, S has no zero. Therefore, if S is unruly, it satisfies conditions (1)–(3) of our corollary.

Conversely, if S satisfies (1)–(3), we need to prove that $\mu(H_e) = |H_e|$ for every maximal subgroup H_e and that all idempotents of S are join-irreducible. Indeed, $\mu(H_e) = |H_e|$ for trivial groups H_e . If H_e belongs to one of the types (i) or (ii), then it contains the least nontrivial subgroup. This subgroup is normal, and hence it is contained in the kernel of every noninjective homomorphism of H_e . Since all structural homomorphisms of H_e are not injective, this least subgroup is contained in all of them, and so it is contained in S_e . This shows that S_e is not trivial, and hence e is join-irreducible. Also, it is easy to check that $\mu(H_e) = |H_e|$ in this case. If e is the least idempotent of S , then $S_e = H_e \neq \{e\}$, and hence e is join-irreducible and $\mu(H_e) = |H_e|$. Now, if H_e is Klein's four-group, then all structural homomorphisms of H_e are trivial, and hence $H_e = S_e$, and again e is join-irreducible and $\mu(H_e) = |H_e|$. Thus $\mu(H_e) = |H_e|$ for all maximal subgroups H_e , and every idempotent that is not isolated is join-irreducible. If an isolated idempotent e is not join-irreducible, it is the l.u.b. of two of its strict minorants. Indeed, e is the l.u.b. of a set $\{f_1, \dots, f_k\}$ of elements. Choose a maximal element e_i in each of the sets $\{t \in S : f_i \leq t < e\}$. If all e_i are equal, then e cannot be the l.u.b. of f_i , and hence $e_i \neq e_j$ for some $i \neq j$. If $e_i \leq s$ and $e_j \leq s$ for some $s \in S$, then $e_i \leq ss^{-1}e$ and $e_j \leq ss^{-1}e$. If both inequalities are equalities, then $e_i = e_j$, contrary to our assumption. Thus one of these inequalities is strict, and, by the maximality of e_i and e_j , we obtain $ss^{-1}e = e$, whence $e \leq ss^{-1}$. It follows that $es \in H_e$, so that $es = e$ and $e \leq s$. Thus e is the l.u.b. of e_i and e_j , which contradicts condition (3). Therefore, e is joint-irreducible, and hence S is unruly. \square

As we have seen in the proof of Corollary 1, if S is a semilattice of groups and e is a join-irreducible idempotent of S , then S_e is an intersection of the kernels of structural homomorphisms $\varphi_{e,f}$ for all $f < e$. Applying this remark to our theorem, we immediately obtain a formula for the minimal degree of a finite semilattice of groups that appeared in [2].

An inverse semigroup S is called *noble* if it has a faithful transitive representation by one-to-one partial transformations of a set. For example, all subdirectly irreducible inverse semigroups are noble, as it is easy to prove. Minimal

nonzero idempotents of S are called *primitive*. It is known [8] that if S is a finite noble inverse semigroup, then its subgroups H_e and H_f are isomorphic for any two primitive idempotents e and f .

Corollary 2 [9]. *If S is a finite noble inverse subgroup, then $\delta(S) = \pi(S) \cdot \delta(G)$, where $\pi(S)$ is the number of primitive idempotents in S and G any maximal subgroup of S which contains a primitive idempotent.*

Proof. As proved in [8], a finite noble inverse semigroup S has the only join-irreducible \mathcal{D} -class D . It contains all primitive idempotents of S , and hence $\iota(D) = \pi(D)$. It remains to observe that if S has no zero, then all idempotents of D are minimal, and hence $S_e = H_e$ for every $e \in D$. If S has a zero, $S_e = H_e$. Thus $\mu(D) = \delta(H_e)$. It remains to apply our Theorem. \square

An inverse semigroup S is *fundamental* if $(\forall e \in E)[ses^{-1} = tet^{-1}] \Rightarrow s = t$ for all $s, t \in S$.

Corollary 3 [3]. *If S is a fundamental inverse semigroup, then $\delta(S)$ is the number of join-irreducible idempotents in S . Moreover, idempotents are join-irreducible in the semilattice E of idempotents of S if they are join-irreducible in S .*

Proof. Clearly, $S_f = \{f\}$ for every $f \in E$. Indeed, if $h \in S_f$, then $heh^{-1} = hfef^{-1} = fef^{-1}$ for every $e \in E$, for if $fe < f$ then $fe < h$, so that $hfef^{-1} = feh^{-1} = (hfe)^{-1} = fe = fef^{-1}$; and if $fe = f$ then $hfef^{-1} = hfh^{-1} = f = fef^{-1}$. Thus $h = f$. It follows that $\mu(D) = 1$ for every join-irreducible \mathcal{D} -class D . Our Theorem implies that $\delta(S) = \sum \{\iota(D) : D \in \mathbf{D}\}$. But $\sum \{\iota(D) : D \in \mathbf{D}\}$ is merely the number of join-irreducible idempotents of S . To prove the last statement suppose that e is join-irreducible in S but not in E . Then e is the l.u.b. of its strict minorants f_1, \dots, f_k in E . If $f_i \leq s$ for all i and some $s \in S$, then $f_i \leq ess^{-1}$ for all i , and hence $e \leq ess^{-1}$, so that $e = ess^{-1}$. Thus $es \in S_e = \{e\}$, and hence $es = e$ and $e \leq s$. Therefore, e is the l.u.b. of f_1, \dots, f_k in S . \square

An inverse semigroup S is called *E -unitary* if $e \leq s$ implies $s \in E$ for all $e \in E$ and $s \in S$. Let K denote the Suschkewitsch kernel (that is, the least ideal) of S . Obviously, $K = H_e$, where e is the least idempotent of S .

Corollary 4. *If S is an E -unitary inverse semigroup, then $\delta(S) = \delta(E) + \delta(K)$.*

Proof. Let e be a join-irreducible idempotent of S . Consider S_e . If e is the least idempotent of S , then $S_e = K$, $\mu(D_e) = \max\{1, \delta(K)\}$ and $\iota(D_e) = 1$. Observe that e is join-irreducible if and only if K is not trivial. Otherwise, $K = \{e\}$ and $\delta(K) = 0$. If e is not the least idempotent, then $f < e$ for some $f \in E$, and so $f < h$ for every $h \in S_e$. Therefore, $h \in E$, so that $S_e = \{e\}$ and $\mu(D_e) = 1$. By our Theorem, $\delta(S) = \sum \iota(D) + \delta(K)$, where the summation is over all join-irreducible \mathcal{D} -classes D except K . Thus, $\sum \iota(D)$ is the number of all join-irreducible idempotents of S , except the least one. Every element of E , except the least one, is join-irreducible in E if and only if it is join-irreducible in S . It follows that $\delta(E) = \sum \iota(D)$. \square

Corollary 5. $\delta(S \times T) = \delta(S) + \delta(T)$.

Proof. First, a few trivial remarks. Let a and b be the least idempotents of S and T , respectively. Then $E_{S \times T} = E_S \times E_T$ and an idempotent is join-irreducible in $S \times T$ exactly when it is of the forms (e, b) or (a, f) with

e and f join-irreducible in S and T , respectively. If $(s, t) \in S \times T$, then $D_{(s, t)} = D_s \times D_t$, and hence a \mathcal{D} -class of $S \times T$ is join-irreducible if and only if it is of the forms $D_s \times H_b$ or $H_a \times D_t$, where D_s and D_t are join-irreducible \mathcal{D} -classes of S and T , respectively. Also, $S_{(e, b)} = S_e \times \{b\}$ and $S_{(a, f)} = \{a\} \times S_f$ for all join-irreducible $e \in E_S$ and $f \in E_T$.

Secondly, prove that $\mu(D_{(e, b)}) = \mu(D_e)$ and $\mu(D_{(a, f)}) = \mu(D_f)$. Let P be a representation of $S_{(e, b)}$ in $H_{(e, b)}$. Then P is a sum of transitive representations corresponding to subgroups F_i of $H_{(e, b)}$. Define $\text{pr}_1 F_i = \{h \in H_e : (\exists t \in T)(h, t) \in F_i\}$. Let $(s, b) \in S_{(e, b)}$. Then $P(s, b) = P(e, b)$ exactly if $(x, y)^{-1}(s, b)(x, y)^{-1} \in F_i$ for all $(x, y) \in H_{(e, b)} = H_e \times K_T$ and all i . Therefore,

$$(x, y)^{-1}(s, b)(x, y)^{-1} = (x^{-1}sx, y^{-1}by) = (x^{-1}sx, b),$$

and so

$$P(s, b) = P(e, b) \Leftrightarrow (\forall i)(\forall x \in H_e)[x^{-1}sx \in \text{pr}_1 F_i].$$

Clearly, $\text{pr}_1 F_i$ is a subgroup of H_e . If we replace each F_i by $\text{pr}_1 F_i \times K_T$, we obtain a representation R of $S_{(e, b)}$ in $H_{(e, b)}$, and

$$P(s, b) = P(e, b) \Leftrightarrow R(s, b) = R(e, b) \Leftrightarrow R_S(s) = R_S(e),$$

where R_S is a representation of S_e in H_e determined by subgroups $\text{pr}_1 F_i$ of H_e . Clearly, $\delta(R_S) = \delta(R) \leq \delta(P)$. Thus if P is a minimal representation of $S_{(e, b)}$ in $H_{(e, b)}$, then R_S is the minimal representation of S_e in H_e , and hence $\mu(D_{(e, b)}) = \mu(D_e)$. The equality $\mu(D_{(a, f)}) = \mu(D_f)$ is proved analogously.

By our Theorem,

$$\begin{aligned} \delta(S \times T) &= \sum \{\iota(D) \cdot \mu(D) : D \in \mathbf{D}_{S \times T}\} \\ &= \sum \{\iota(D_{(e, b)}) \cdot \mu(D_{(e, b)}) : D_e \in \mathbf{D}_S\} \\ &\quad + \sum \{\iota(D_{(a, f)}) \cdot \mu(D_{(a, f)}) : D_f \in \mathbf{D}_T\} \\ &= \sum \{\iota(D_e) \cdot \mu(D_e) : D_e \in \mathbf{D}_S\} + \sum \{\iota(D_f) \cdot \mu(D_f) : D_f \in \mathbf{D}_T\} \\ &= \delta(S) + \delta(T). \quad \square \end{aligned}$$

Problems. (1) Study $\delta_G(H)$, where H is a (normal) subgroup of a group G .

(2) Suppose that a finite inverse semigroup S is a subdirect product of inverse semigroups $\{S_i : 1 \leq i \leq n\}$. Assume that this decomposition is irredundant (no S_i can be omitted). Then $\delta(S) \leq \delta(S_1) + \delta(S_2) + \cdots + \delta(S_n)$. When is this inequality an equality? Can it be strict?

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE,
ARKANSAS 72701

E-mail address: bschein@uafsysb.bitnet

E-mail address: bschein@uark.edu.internet